Alexandrov Existence and Uniqueness Theorem for compact surfaces

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Original statement for the sphere

A.D. Alexandrov, 1942

Let $m(0,+)$ be a flat (0-curvature) metric on the sphere $\mathbb{S}^2$ with conical singularities of positive singular curvature. Then

1. There exists a convex polytope P in $\mathbb{R}^3$ such that P *realizes* $m(0,+)$ (i.e. the induced metric on P is isometric to $m(0,+)$).
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2. A $(-1,+)$-metric the sphere is realized by a unique (up to congruence) convex polytope in the 3-d hyperbolic space.

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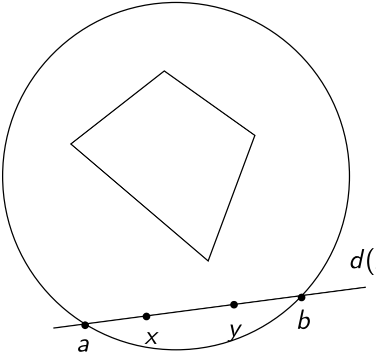

$$d(x, y) = \frac{1}{2} \ln([a, b, x, y])$$

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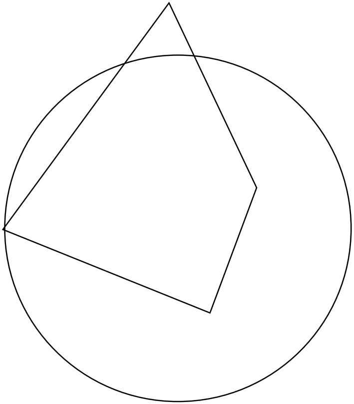

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And infinitesimal rigidity is a projective property!

Darboux, Sauer, Knebelman 1930 , Pogorelov, Volkov

Let (M, g) and $(M, \overline{g})$ be two (pseudo-)Riemannian manifolds and f a map sending geodesics of g to geodesics of $\overline{g}$. Let $\lambda = |\overline{g}|/|g|$. Let K be a Killing field of (M, g). Then the vector field $\overline{K}$ of $(M, \overline{g})$ defined by

$$\overline{g}(\overline{K}, X) = g(\lambda^{\frac{1}{n+1}} K, X), \quad \forall X \in TM,$$

is a Killing field of $(M, \overline{g})$.

Interlude on projective nature of infinitesimal rigidity

The proof is a simple consequence of the Weyl equation for connections having the same geodesics

$$\overline{\nabla}_X Y - \nabla_X Y = \frac{1}{2}\frac{1}{n+1}(X.\ln\lambda)Y + \frac{1}{2}\frac{1}{n+1}(Y.\ln\lambda)X.$$

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There is another interpretation based on infinitesimal rigidity of frameworks (see Ivan Izmestiev).

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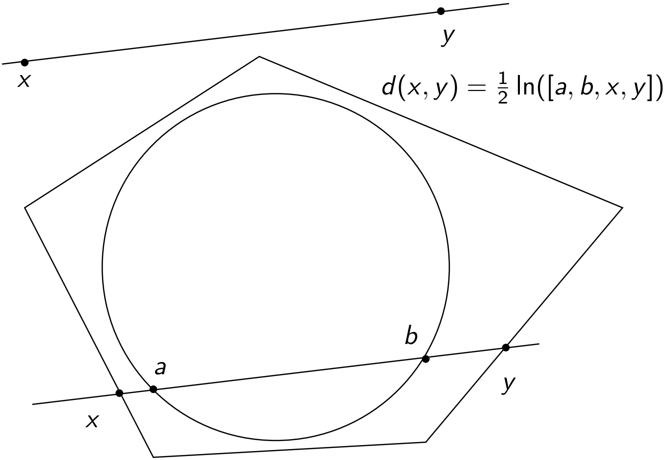

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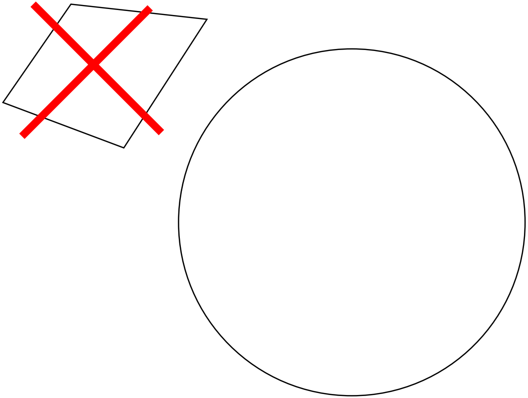

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Let $m(1,-)$ be a spherical metric on the sphere with conical singularities of negative singular curvature [and length of contractible geodesics $> 2\pi$]. Then

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By projective duality, this theorem is equivalent to a realization theorem for hyperbolic polyhedra with prescribed Gauss image.

Higher genus?

- Due to Gauss–Bonnet formula, there does not exist other (K, ϵ)-metric on the sphere, $K \in \{-1, 0, 1\}, \epsilon \in \{-, +\}$, than $(0, +), (1, +), (-1, +), (1, -)$.

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- We will realize the universal cover of the metric, as a convex polyhedral surface equivariant under the action of the fundamental group.

Example of flat surfaces of genus > 1

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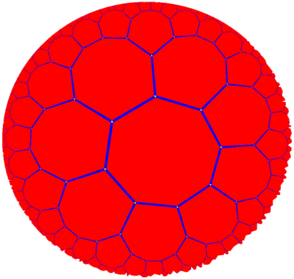

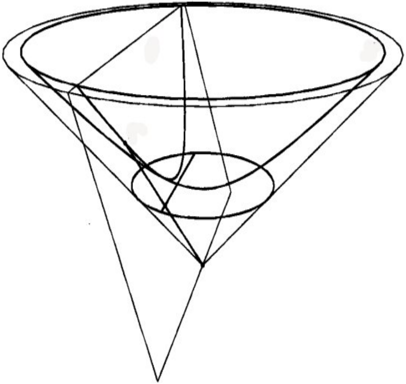

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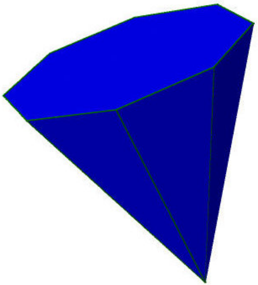
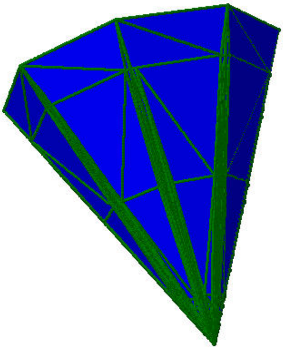
The quotient of this polyhedral surface is a compact surface of genus 2 with a flat metric with one conical singularity of curvature $2\pi - 8 \times \frac{3\pi}{4} = -4\pi < 0$ (cone-angle 6π).





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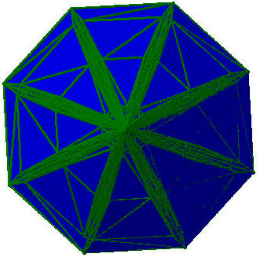

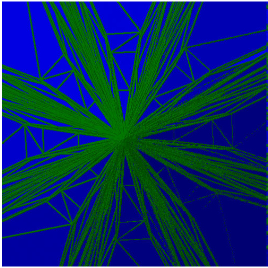

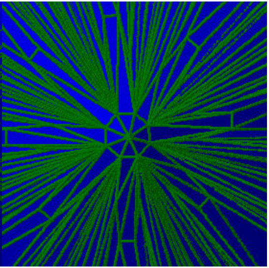

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The quotient is a torus with a hyperblic metric with one conical singularity of positive curvature.

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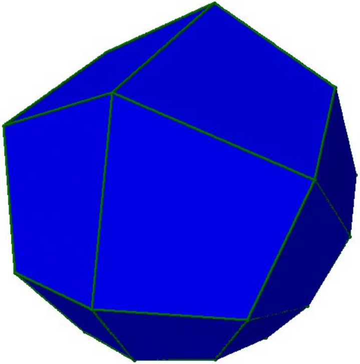

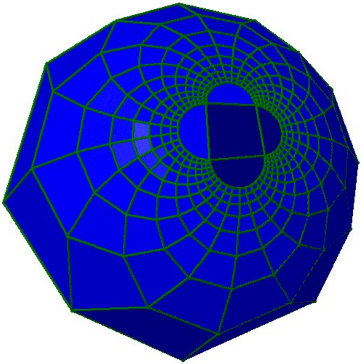

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By Gauss–Bonnet, there are only 10 possible (K, ϵ, g).

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Alexandrov Theorem for compact surfaces

Let m be a (K, ϵ)-metric on S (large for $(1, -)$). Then

1. There exists a convex polyhedral surface $P \subset M_K^\epsilon$ and a group of isometries Γ, acting cocompatly on a totally umbilic surface, such that P/Γ is isometric to (S, m).
2. (P, Γ) is the unique couple of this kind, up to congruences.

Благодарю вас за внимание